J. Math. Anal. Appl. 356 (2009) 418–428

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiple solutions for discrete boundary value problems

Alberto Cabada [a,1], Antonio Iannizzotto [b,*], Stepan Tersian [c]

[a] Departamento de Analise Matematica, Facultade de Matematicas, Universitade de Santiago de Compostela, Santiago de Compostela, Spain
[b] Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria 6, 95125 Catania, Italy
[c] Department of Mathematical Analysis, University of Rousse, 7017 Rousse, Bulgaria

A R T I C L E   I N F O

Article history:
Received 2 October 2008
Available online 28 February 2009
Submitted by V. Radulescu

Keywords:
Difference equations
Discrete p-Laplacian
Variational methods

A B S T R A C T

A recent multiplicity theorem for the critical points of a functional defined on a finite-dimensional Hilbert space, established by Ricceri, is extended. An application to Dirichlet boundary value problems for difference equations involving the discrete p-Laplacian operator is presented.

1. Introduction

In the present paper we will deal with the following boundary value problem, with homogeneous Dirichlet conditions, for a difference equation, depending on a real parameter λ:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p x(k-1) = \lambda f\big(k, x(k)\big) & \text{for every } k \in [1, T], \\ x(0) = x(T+1) = 0, \end{cases}$$

where $T \geqslant 2$ is an integer, $[1, T]$ denotes the discrete interval $\{1, 2, \ldots, T\}$, $p > 1$ is a real number, Δ_p is the *discrete p-Laplacian operator* defined by

$$\Delta_p x(k-1) = \Delta\big[\big|\Delta x(k-1)\big|^{p-2}\Delta x(k-1)\big]$$

(Δ denotes the forward difference operator) and f is a continuous function defined on $[1, T] \times \mathbb{R}$ (see Section 3 below for details).

Under convenient assumptions on the function f, we will prove the existence of a positive λ^* such that the problem (P_{λ^*}) admits at least three solutions (see Theorem 6 below).

Boundary value problems for difference equations have been extensively studied (see the monographs of Lakshmikantham and Trigiante [9] and of Agarwal [1]): the classical theory of difference equations employs numerical analysis and features from the linear and nonlinear operator theory, such as fixed point methods; we remark that, usually, the application of the fixed point methods yields existence results only.

Recently, although, many new results have been established by applying variational methods: we recall here the works of Agarwal, Perera and O'Regan [2,3], Cai, Guo and Yu [4], Cai and Yu [5], Faraci and Iannizzotto [6], Guo and Ma [7], Jiang and Zhou [8], Mihăilescu, Rădulescu and Tersian [10]; the variational approach represents an important advance as it allows to prove multiplicity results as well.

* Corresponding author.
E-mail addresses: alberto.cabada@usc.es (A. Cabada), iannizzotto@dmi.unict.it (A. Iannizzotto), sterzian@ru.acad.bg (S. Tersian).

1 Partially supported by Ministerio de Educación y Ciencia, Spain, project MTM2007-61724, and by Xunta de Galicia, Spain, project PGIDIT06PXIB207023PR.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.02.038

In all the aforementioned papers, discrete boundary value problems involving a variety of operators and boundary conditions are studied in a variational framework: solutions are seen as critical points of a convenient energy functional, defined on a function space; in general, such function spaces have *finite dimension*, which makes things easier (in comparison with the variational methods for differential equations).

In the present paper, we study the problem (P_λ) following a variational approach, based on a recent result of Ricceri (see [12]): such result assures the existence of at least three critical points for a certain class of functionals defined on a *finite-dimensional* normed space.

Thus, Ricceri's result is suitable for applications in the field of difference equations: such application yields a multiplicity result for a discrete boundary value problem of the type (P_λ) involving the discrete Laplacian operator ($p = 2$).

In the present paper, we extend Ricceri's abstract result replacing 2 with an arbitrary real number $p > 1$ (see Theorem 3 below), apply it to the case of the p-Laplacian (see Theorem 6 below), and provide some new information about the intrinsic properties of the function space involved: namely, we establish the precise embedding constants of the function space involved into the space $\mathbb{R}^T$ with the maximum norm (see Lemma 4 below), improving a previous result of Jiang and Zhou [8].

The paper is organized as follows: in Section 2 we state and prove our abstract result; in Section 3 we apply it to the problem (P_λ); in Section 4 we discuss some limit cases and give examples.

2. The abstract result

Before introducing our result, let us recall, for the convenience of the reader, a recent theorem of Ricceri (see [12, Theorem A] or [11, Theorem 1]) which will be employed in our proof.

Theorem 1. *Let (X, τ) be a Hausdorff space and $\Phi, J : X \to \mathbb{R}$ be functionals; moreover, let M be the (possibly empty) set of all the global minimizers of J and define*

$$\alpha = \inf_{x \in X} \Phi(x),$$
$$\beta = \begin{cases} \inf_{x \in M} \Phi(x) & \text{if } M \neq \emptyset, \\ \sup_{x \in X} \Phi(x) & \text{if } M = \emptyset. \end{cases}$$

Assume that the following conditions are satisfied:

(1.1) *for every $\mu > 0$ and every $\rho \in \mathbb{R}$ the set $\{x \in X\colon \Phi(x) + \mu J(x) \leqslant \rho\}$ is sequentially compact (if not empty);*
(1.2) $\alpha < \beta$.

Then, at least one of the following conditions holds:

(1.3) *there exists a continuous mapping $h : (\alpha, \beta) \to X$ with the following property: for every $t \in (\alpha, \beta)$, one has*

$$\Phi\big(h(t)\big) = t$$

and for every $x \in \Phi^{-1}(t)$, $x \neq h(t)$,

$$J(x) > J\big(h(t)\big);$$

(1.4) *there exists $\mu^* > 0$ such that the functional $\Phi + \mu^* J$ admits at least two global minimizers in X.*

We will also use the following consequence of the finite-dimensional version of the Mountain Pass Theorem (see Struwe [13, p. 74]): let $C^1(X, \mathbb{R})$ denote the set of all Gâteaux differentiable functions defined on X, whose derivatives are continuous in X.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space, $\dim(X) < \infty$, and $E \in C^1(X, \mathbb{R})$ be a coercive functional having at least two strict local minimizers $x_0, x_1 \in X$. Then, E has a critical point $x_2 \in X \setminus \{x_0, x_1\}$.*

Now we can introduce our abstract result, which is a simple extension of the main result of Ricceri [12]: here, an arbitrary real number $p > 1$ replaces 2 (we include the proof for the sake of completeness).

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space, $\dim(X) < \infty$, $J \in C^1(X, \mathbb{R})$, $\bar{x} \in X$ and $p, r, s \in \mathbb{R}$ with $p > 1$, $0 < r < s$. Assume that the functional $x \mapsto \|x\|^p$ is continuously Gâteaux differentiable in X and that the following conditions are satisfied:*

(3.1) $\liminf_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^p} \geqslant 0$;
(3.2) $\inf_{x \in X} J(x) < \inf_{\|x - \bar{x}\| \leqslant s} J(x)$;
(3.3) $J(\bar{x}) \leqslant \inf_{r \leqslant \|x - \bar{x}\| \leqslant s} J(x)$.

Then, there exists $\lambda^* > 0$ *such that the functional*

$$x \mapsto \frac{\|x-\bar{x}\|^p}{p} + \lambda^* J(x)$$

admits at least three critical points in X.

Proof. We are going to apply Theorem 1: so, we denote by τ the norm topology on X and define a continuous functional Φ by putting for every $x \in X$

$$\Phi(x) = \begin{cases} \|x-\bar{x}\|^p & \text{if } \|x-\bar{x}\| < r, \\ r^p & \text{if } r \leqslant \|x-\bar{x}\| \leqslant s, \\ \|x-\bar{x}\|^p - s^p + r^p & \text{if } s < \|x-\bar{x}\|. \end{cases}$$

First, we prove the inequality

$$\beta > r^p, \tag{1}$$

distinguishing two cases:

- if $M \neq \emptyset$, since M is closed and Φ is coercive, there is some $\bar{y} \in M$ such that $\Phi(\bar{y}) = \beta$, which, by (3.2), implies that $\|\bar{y}-\bar{x}\| > s$, in particular
$$\beta = \|\bar{y}-\bar{x}\|^p - s^p + r^p > r^p;$$
- if $M = \emptyset$, clearly $\beta = +\infty$.

Now we prove that all the assumptions of Theorem 1 hold in the present case, starting with (1.1): by (3.1) we get for every $\mu > 0$

$$\lim_{\|x\| \to +\infty} \left[\Phi(x) + \mu J(x)\right] = +\infty,$$

which implies that for every $\rho \in \mathbb{R}$ the corresponding sublevel set of $\Phi + \mu J$ is bounded and closed; hence, such set is (sequentially) compact, if not empty.

In order to prove that (1.2) is satisfied, we observe that

$$\inf_{x \in X} \Phi(x) = 0$$

and we invoke (1).

By Theorem 1, either (1.3) or (1.4) holds: actually, we will prove that (1.4) is true, arguing by contradiction.

Assume that (1.4) is false: then, (1.3) must be satisfied, so let the continuous mapping $h : (0, \beta) \to X$ be defined as above; by using (1), it is easily seen that

$$\left\|h(t) - \bar{x}\right\| < r \quad \text{iff} \quad t < r^p,$$
$$r \leqslant \left\|h\left(r^p\right) - \bar{x}\right\| \leqslant s,$$
$$\left\|h(t) - \bar{x}\right\| > s \quad \text{iff} \quad t > r^p,$$

which contradicts the continuity of h.

By (1.4), there exists $\mu^* > 0$ such that the functional $\Phi + \mu^* J$ has at least two global minimizers $x_0, x_1 \in X$ ($x_0 \neq x_1$): we prove that

$$\|x_i - \bar{x}\| < r \quad \text{or} \quad \|x_i - \bar{x}\| > s \quad (i = 0, 1), \tag{2}$$

arguing again by contradiction; indeed, if $r \leqslant \|x_i - \bar{x}\| \leqslant s$, by (3.3) we obtain

$$\Phi(x_i) + \mu^* J(x_i) = r^p + \mu^* J(x_i) > \mu^* J(x_i) \geqslant \mu^* J(\bar{x}) = \Phi(\bar{x}) + \mu^* J(\bar{x}),$$

against the fact that x_i is a global minimizer for $\Phi + \mu^* J$.

Set

$$\lambda^* = \frac{\mu^*}{p}.$$

From (2) and the definition of Φ it follows that both x_0 and x_1 are *local* minimizers of the functional $E \in C^1(X, \mathbb{R})$ defined for all $x \in X$ by putting

$$E(x) = \frac{\|x-\bar{x}\|^p}{p} + \lambda^* J(x).$$

We prove that E has at least one critical point $x_2 \in X \setminus \{x_0, x_1\}$, considering two cases:

- if both x_0, x_1 are *strict* local minimizers of E, an application of Theorem 2 gives the desired result;
- if either x_0 or x_1 is not a strict local minimizer, E obviously admits infinitely many local minimizers (in particular, critical points) at the same level.

Thus, the proof is complete. $\square$

Some comments are now in order: in the proof, we have used the fact that X has finite dimension (in proving (1.1)); Ricceri has shown that, if the dimension of X is infinite, the conclusion of Theorem 3 does not hold for $p = 2$ (see [12, Example 1 and Remark 1] for a further discussion about possible extensions to the infinite-dimensional case).

In particular, our abstract result has no direct application to variational problems involving infinite-dimensional Banach spaces (such as boundary value problems for differential equations).

Finally, we remark that the hypothesis that the functional $x \mapsto \|x\|^p$ is continuously Gâteaux differentiable is here essential: such hypothesis does not hold in general (for instance, consider the case $X = \mathbb{R}^2$ with the maximum norm and an arbitrary $p > 1$), but it holds in most applications (see Lemma 5 below).

3. An application

In the present section we are going to apply Theorem 3 to the problem (P_λ) introduced in Section 1: namely, we will prove that, under convenient assumptions on the function f, there exists $\lambda^* > 0$ such that (P_{λ^*}) admits at least three solutions.

We need to introduce some notation: first of all, for every $a, b \in \mathbb{Z}$, $a \leqslant b$, we define the discrete interval

$$[a, b] = \{a, a+1, \dots, b\}.$$

Let $T \in \mathbb{N}$, $T \geqslant 2$ and $p \in \mathbb{R}$, $p > 1$: we will deal with functions $x : [0, T+1] \to \mathbb{R}$, for which we introduce the *forward difference operator* Δ by putting for every $k \in [1, T+1]$

$$\Delta x(k-1) = x(k) - x(k-1);$$

moreover, we introduce for every real $\gamma > 1$ the mapping $\varphi_\gamma : \mathbb{R} \to \mathbb{R}$ by putting for every $t \in \mathbb{R}$

$$\varphi_\gamma(t) = |t|^{\gamma-2} t$$

and, for any $p > 1$, the *discrete p-Laplacian operator* Δ_p defined by

$$\Delta_p x(k-1) = \Delta\varphi_p\big(\Delta x(k-1)\big).$$

Finally, let $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ be such that $f(k, \cdot)$ is continuous for every $k \in [1, T]$.

The solutions of the boundary value problem (P_λ) (for an arbitrary $\lambda > 0$) can be found as elements of a convenient function space: we define the real vector space

$$X = \big\{x : [0, T+1] \to \mathbb{R}\colon\ x(0) = x(T+1) = 0\big\}$$

and for every $x \in X$ we denote

$$\|x\| = \left[\sum_{k=1}^{T+1} \big|\Delta x(k-1)\big|^p\right]^{\frac{1}{p}},$$

so $(X, \|\cdot\|)$ is a Banach space and $\dim(X) = T$; we also put for every $x \in X$

$$\|x\|_\infty = \max_{k \in [1,T]} \big|x(k)\big|.$$

By classical results, the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent on X: the following lemma yields the precise constants determining the relation between the two.

Denote

$$c_1 = \begin{cases} [(\frac{2}{T})^{p-1} + (\frac{2}{T+2})^{p-1}]^{\frac{1}{p}} & \text{if } T \text{ is even},\\ \frac{2}{(T+1)^{\frac{p-1}{p}}} & \text{if } T \text{ is odd} \end{cases}$$

and

$$c_2 = \big[2 + 2^p(T-1)\big]^{\frac{1}{p}}.$$

Lemma 4. *Let*

$$S = \big\{x \in X\colon\ \|x\|_\infty = 1\big\}.$$

Then, the following conditions hold:

(4.1) $\min_{x\in S}\|x\| = c_1$;
(4.2) $\max_{x\in S}\|x\| = c_2$.

Proof. First, we observe that the set S is compact.

We prove (4.1): by compactness, there exists $x \in S$ which minimizes $\|\cdot\|$ over S; there is, also, $\tau \in [1, T]$ such that $|x(\tau)| = 1$ and $|x(k)| < 1$ for every $k \in [0, \tau - 1]$; without any loss of generality, we may assume that $x(\tau) = 1$.

Next, we will deduce from the minimality property of x some information about the geometry of such function.

We prove that

$$x(k-1) \leqslant x(k) \quad \text{for every } k \in [1, \tau], \tag{3}$$

arguing by contradiction: indeed, let $h \in [1, \tau]$ be such that $x(h-1) > x(h)$; then, clearly $h \leqslant \tau - 1$ and there is some $j \in [h, \tau - 1]$ fulfilling

$$x(j) \leqslant x(h-1) \leqslant x(j+1);$$

hence, we define $y \in S$ by putting

$$y(k) = \begin{cases} x(k) & \text{if } k \in [0, h-1], \\ x(h-1) & \text{if } k \in [h, j], \\ x(k) & \text{if } k \in [j+1, T+1] \end{cases}$$

and we get

$$\begin{aligned} \|x\|^p - \|y\|^p &= \sum_{k=1}^{T+1}\left[\left|\Delta x(k-1)\right|^p - \left|\Delta y(k-1)\right|^p\right] \\ &= \sum_{k=h}^{j}\left|\Delta x(k-1)\right|^p + \left[x(j+1) - x(j)\right]^p - \left[x(j+1) - x(h-1)\right]^p \\ &\geqslant \sum_{k=h}^{j}\left|\Delta x(k-1)\right|^p > 0, \end{aligned}$$

which implies $\|y\| < \|x\|$, a contradiction.

An analogous argument leads to the following relation:

$$x(k-1) \geqslant x(k) \quad \text{for every } k \in [\tau + 1, T + 1]. \tag{4}$$

We can obtain more precise information:

$$x(k) = \begin{cases} \frac{k}{\tau} & \text{if } k \in [0, \tau], \\ \frac{T+1-k}{T+1-\tau} & \text{if } k \in [\tau + 1, T + 1]. \end{cases} \tag{5}$$

Indeed, we already know that $x(0) = x(T+1) = 0$ and $x(\tau) = 1$; moreover, relations (3) and (4) lead us to solve two constrained minimization problems:

- first, we put $z_k = \Delta x(k-1)$ for every $k \in [1, \tau]$ and consider the problem $\min_{z\in Q} \psi(z)$, where $z = (z_1, \ldots, z_\tau)$ and

  $$Q = \left\{z \in \mathbb{R}^\tau\colon\ 0 \leqslant z_k \leqslant 1,\ \sum_{k=1}^{\tau} z_k = 1\right\}, \qquad \psi(z) = \sum_{k=1}^{\tau} z_k^p;$$

  by the elementary inequality

  $$\frac{1}{\tau}\sum_{k=1}^{\tau} z_k \leqslant \left(\frac{1}{\tau}\sum_{k=1}^{\tau} z_k^p\right)^{\frac{1}{p}},$$

  where the equality holds for

  $$z_1 = z_2 = \cdots = z_\tau = \frac{1}{\tau},$$

  it follows that

  $$\min_{z\in Q}\psi(z) = \psi\left(\frac{1}{\tau}, \ldots, \frac{1}{\tau}\right) = \frac{1}{\tau^{p-1}};$$

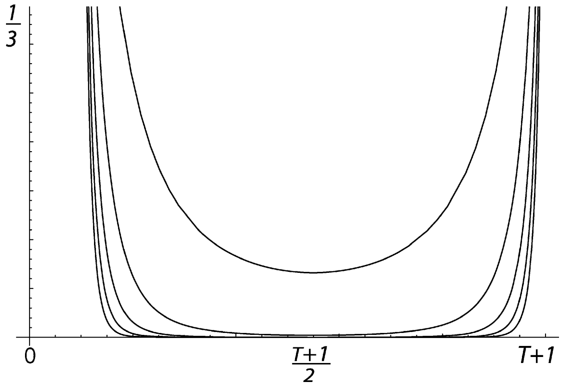

Fig. 1. Graph of the function $\xi_{T,p}$ for $T = 10$, $1 \leqslant t \leqslant 10$ and $p = 2, 4, \dots, 10$.

- analogously, we get for every $k \in [\tau + 1, T + 1]$

$$\Delta x(k-1) = -\frac{1}{T+1-\tau}.$$

The above equalities imply (5).

Thus, we obtain

$$\|x\|^p = \sum_{k=1}^{\tau} \frac{1}{\tau^p} + \sum_{k=\tau+1}^{T+1} \frac{1}{(T+1-\tau)^p} = \frac{1}{\tau^{p-1}} + \frac{1}{(T+1-\tau)^{p-1}}.$$

We still need to find τ: with this aim in mind, we observe that the function $\xi_{T,p} : (0, T+1) \to \mathbb{R}$ defined by

$$\xi_{T,p}(t) = \frac{1}{t^{p-1}} + \frac{1}{(T+1-t)^{p-1}}$$

attains its minimum at $t = \frac{T+1}{2}$, while the same function is decreasing in $(0, \frac{T+1}{2})$ and increasing in $(\frac{T+1}{2}, T+1)$, see Fig. 1.

Now we distinguish two cases:

- if T is even, we choose $\tau = \frac{T}{2}$ or, equivalently, $\tau = \frac{T+2}{2}$ and get

$$\|x\| = \left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]^{\frac{1}{p}};$$

- if T is odd, we choose $\tau = \frac{T+1}{2}$ and get

$$\|x\| = \frac{2}{(T+1)^{\frac{p-1}{p}}}.$$

This proves (4.1).

Now we prove (4.2): given $x \in S$, we observe that

$$\|x\|^p = \left|x(1)\right|^p + \sum_{k=2}^{T} \left|\Delta x(k-1)\right|^p + \left|x(T)\right|^p \leqslant 2 + 2^p(T-1),$$

so

$$\max_{x \in S} \|x\| \leqslant c_2;$$

on the other hand, we may define $x \in S$ by putting $x(k) = (-1)^k$ for every $k \in [1, T]$ and get

$$\|x\|^p = 2 + 2^p(T-1),$$

which implies (4.2) and concludes the proof. $\square$

Lemma 4 above represents a refined version of Lemma 2.2 of Jiang and Zhou [8].

A variational framework for problem (P_λ) is provided as follows: for every $k \in [1, T]$ and every $t \in \mathbb{R}$ we put

$$F(k,t) = \int_0^t f(k,\tau)\,d\tau,$$

for every $x \in X$

$$J(x) = -\sum_{k=1}^{T} F\big(k, x(k)\big)$$

and for every $\lambda > 0$ and every $x \in X$

$$E_\lambda(x) = \frac{\|x\|^p}{p} + \lambda J(x).$$

Lemma 5. *For every $\lambda > 0$, E_λ is continuously Gâteaux differentiable, and for every $x, y \in X$*

(5.1) $\langle E'_\lambda(x), y\rangle = -\sum_{k=1}^{T}[\Delta_p x(k-1) + \lambda f(k, x(k))]y(k)$.

Proof. Clearly $E_\lambda \in C^1(X,\mathbb{R})$; in what follows we prove (5.1): choose $x, y \in X$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an arbitrary mapping: we recall the *summation by parts formula*

$$\sum_{k=1}^{T}\big[\varphi\big(\Delta x(k-1)\big)\Delta y(k-1) + \Delta\varphi\big(\Delta x(k-1)\big)y(k)\big] = \varphi\big(\Delta x(T)\big)y(T). \tag{6}$$

Using (6) with $\varphi = \varphi_p$, we get

$$\begin{aligned}\lim_{\delta\to 0^+}\frac{\|x+\delta y\|^p - \|x\|^p}{p\delta} &= \sum_{k=1}^{T+1}\varphi_p\big(\Delta x(k-1)\big)\Delta y(k-1)\\ &= -\sum_{k=1}^{T}\Delta\varphi_p\big(\Delta x(k-1)\big)y(k).\end{aligned}$$

Besides, we have

$$\lim_{\delta\to 0^+}\frac{J(x+\delta y) - J(x)}{\delta} = -\sum_{k=1}^{T} f\big(k, x(k)\big)y(k).$$

The equalities above imply (5.1). $\square$

Now we can introduce our multiplicity result for the solutions of the problem (P_λ).

Theorem 6. *Let T, p, f, F be as above and $r, s \in \mathbb{R}$ satisfy $0 < r < s$. Moreover, assume that the following conditions hold*:

(6.1) $\limsup_{|t|\to+\infty}\frac{F(k,t)}{|t|^p} \leqslant 0$ *for every* $k \in [1, T]$;
(6.2) $\sum_{k=1}^{T}\sup_{|t|\leqslant\frac{s}{c_1}} F(k,t) < \sum_{k=1}^{T}\sup_{t\in\mathbb{R}} F(k,t)$;
(6.3) $\sup_{\frac{r}{c_2}\leqslant|t|\leqslant\frac{s}{c_1}} F(k,t) \leqslant -\sum_{h\neq k}\sup_{|t|\leqslant\frac{s}{c_1}} F(h,t)$ *for every* $k \in [1, T]$.

Then, there exists $\lambda^* > 0$ such that (P_{λ^*}) admits at least three solutions.

Proof. We are going to apply Theorem 3 with X, J, p defined above and $\bar{x} = 0$: hence, we need to check that all hypotheses of that result are satisfied.

We prove that (3.1) holds: since X has finite dimension, there exists $c > 0$ such that for every $x \in X$

$$\|x\| \geqslant c\left[\sum_{k=1}^{T}\big|x(k)\big|^p\right]^{\frac{1}{p}}.$$

Choose $\varepsilon > 0$: by (6.1), there exists $K > 0$ such that for every $k \in [1, T]$ and $t \in \mathbb{R}$ with $|t| > K$

$$\frac{F(k,t)}{|t|^p} < \frac{c^p\varepsilon}{T};$$

let

$$M = \max_{k\in[1,T],\, |t|\leqslant K} \left|F(k,t)\right|,$$

then for every $x \in X$ with

$$\|x\| > \left(\frac{MT}{\varepsilon}\right)^{\frac{1}{p}}$$

we get

$$\frac{J(x)}{\|x\|^p} \geqslant -\sum_{|x(k)|\leqslant K} \frac{\varepsilon}{T} - \sum_{|x(k)|>K} \frac{|F(k,x(k))|}{c^p|x(k)|^p} \geqslant -\varepsilon,$$

which proves (3.1).

We prove that (3.2) holds, distinguishing two cases:

- First, assume that

  $$\inf_{x\in X} J(x) > -\infty.$$

  We prove that for every $\sigma > 0$ the following equality holds:

  $$\inf_{\|x\|_\infty\leqslant\sigma} J(x) = -\sum_{k=1}^{T} \sup_{|t|\leqslant\sigma} F(k,t). \tag{7}$$

  To see this, note that for every $x \in X$, $\|x\|_\infty \leqslant \sigma$ we clearly have

  $$J(x) = -\sum_{k=1}^{T} F\bigl(k,x(k)\bigr) \geqslant -\sum_{k=1}^{T} \sup_{|t|\leqslant\sigma} F(k,t);$$

  on the other hand, for every $\varepsilon > 0$ and every $k \in [1,T]$ there is some $t_k \in \mathbb{R}$, $|t_k| \leqslant \sigma$ such that

  $$F(k,t_k) > \sup_{|t|\leqslant\sigma} F(k,t) - \frac{\varepsilon}{T},$$

  so, defined $\tilde{x} \in X$ by putting $\tilde{x}(k) = t_k$ for every $k \in [1,T]$, we get $\|\tilde{x}\|_\infty \leqslant \sigma$ and

  $$J(\tilde{x}) > -\sum_{k=1}^{T} \sup_{|t|\leqslant\sigma} F(k,t) + \varepsilon,$$

  which proves (7).
  In a similar way, we deduce that

  $$\inf_{x\in X} J(x) = -\sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t). \tag{8}$$

  Then, from (4.1), (6.2), (7) and (8) we deduce that

  $$\inf_{x\in X} J(x) = -\sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t) < -\sum_{k=1}^{T} \sup_{|t|\leqslant\frac{s}{c_1}} F(k,t) = \inf_{\|x\|_\infty\leqslant\frac{s}{c_1}} J(x) \leqslant \inf_{\|x\|\leqslant s} J(x).$$

- Now, assume that

  $$\inf_{x\in X} J(x) = -\infty;$$

  then, the inequality (3.2) is clearly fulfilled.

We prove that (3.3) holds: clearly $J(0) = 0$, while for every $x \in X$ satisfying $r \leqslant \|x\| \leqslant s$ we have by (4.1) and (4.2)

$$\frac{r}{c_2} \leqslant \|x\|_\infty \leqslant \frac{s}{c_1};$$

there exists $k \in [1,T]$ such that $\|x\|_\infty = |x(k)|$, so by (6.3) we get

$$J(x) = -F\bigl(k,x(k)\bigr) - \sum_{h\neq k} F\bigl(h,x(h)\bigr) \geqslant -\sup_{\frac{r}{c_2}\leqslant|t|\leqslant\frac{s}{c_1}} F(k,t) - \sum_{h\neq k} \sup_{|t|\leqslant\frac{s}{c_1}} F(h,t) \geqslant 0.$$

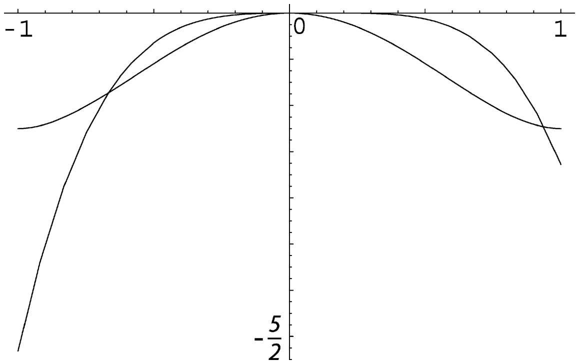

Fig. 2. Graphs of the functions $F(1,\cdot)$ and $F(2,\cdot)$ for $-1 \leqslant t \leqslant 1$.

Thus, by Theorem 3 there exists $\lambda^* > 0$ such that E_{λ^*} admits at least three critical points in X: let us denote them x_0, x_1, x_2.

Finally, we prove that x_i $(i = 0, 1, 2)$ is a solution of (P_{λ^*}): indeed, recalling (5.1), we get for every $y \in X$

$$-\sum_{k=1}^{T}\big[\Delta_p x_i(k-1) + \lambda^* f\big(k, x_i(k)\big)\big]y(k) = 0,$$

which obviously implies that x_i solves (P_{λ^*}). □

4. Remarks and examples

In this final section, we are going to discuss the main features of Theorem 6, presenting some examples in connection.

We start with a simple example of a system complying with all hypotheses of Theorem 6: note that $p \neq 2$, so the case under examination cannot be solved applying the results of [12].

Example 7. Consider the system

$$\begin{cases} -\varphi_5\big(\Delta x(1)\big) + \varphi_5\big(\Delta x(0)\big) = 2\lambda\big(x(1)^3 - x(1)\big), \\ -\varphi_5\big(\Delta x(2)\big) + \varphi_5\big(\Delta x(1)\big) = -4\lambda\left(x(2) - \dfrac{1}{10}\right)^3, \\ x(0) = x(3) = 0, \end{cases} \tag{9}$$

which is of the type (P_λ) with $T = 2$, $p = 5$ and

$$f(1,t) = 2\big(t^3 - t\big), \qquad f(2,t) = -4\left(t - \frac{1}{10}\right)^3,$$

that is,

$$F(1,t) = \frac{t^4}{2} - t^2, \qquad F(2,t) = \frac{1}{10^4} - \left(t - \frac{1}{10}\right)^4$$

(see Fig. 2).

Note that in this case we have $c_1 = (\frac{17}{16})^{\frac{1}{5}}$ and $c_2 = 34^{\frac{1}{5}}$.

By a straightforward computation, we see that the condition (6.1) is fulfilled; moreover, we put $r = \frac{c_2}{5}$ and $s = c_1$ and obtain

$$\sup_{0.2 \leqslant |t| \leqslant 1} F(1,t) = -0.0392, \qquad \sup_{|t| \leqslant 1} F(1,t) = 0, \qquad \sup_{t \in \mathbb{R}} F(1,t) = +\infty$$

and

$$\sup_{0.2 \leqslant |t| \leqslant 1} F(2,t) = 0, \qquad \sup_{|t| \leqslant 1} F(2,t) = \sup_{t \in \mathbb{R}} F(2,t) = 0.0001,$$

which implies conditions (6.2) and (6.3).

Thus, by Theorem 6, there exists $\lambda^* > 0$ such that the system (9) has at least three solutions.

Next, a brief discussion about the main hypotheses of Theorem 6 is in order: indeed, while the condition (6.1) is a standard coercivity assumption, conditions (6.2) and (6.3) are rather unusual; hence, it is a natural question whether such assumptions can be removed or weakened.

The answer is, in general, negative, as the following examples will show.

Example 8. Consider the system

$$\begin{cases} -\Delta x(1) + \Delta x(0) = -\lambda x(1), \\ -\Delta x(2) + \Delta x(1) = \lambda, \\ x(0) = x(3) = 0, \end{cases} \tag{10}$$

which is of the type (P_λ) with $T = p = 2$ and

$$f(1,t) = -t, \qquad f(2,t) = 1.$$

We have then

$$F(1,t) = -\frac{t^2}{2}, \qquad F(2,t) = t.$$

Note that in this case we have $c_1 = (\frac{3}{2})^{\frac{1}{2}}$ and $c_2 = 6^{\frac{1}{2}}$.

It is easily seen that the condition (6.1) is fulfilled; besides, for arbitrary $0 < r < s$ we have

$$\sup_{\frac{r}{c_2} \leqslant |t| \leqslant \frac{s}{c_1}} F(1,t) = -\frac{r^2}{2c_2^2}, \qquad \sup_{|t| \leqslant \frac{s}{c_1}} F(1,t) = \sup_{t \in \mathbb{R}} F(1,t) = 0$$

and

$$\sup_{\frac{r}{c_2} \leqslant |t| \leqslant \frac{s}{c_1}} F(2,t) = \sup_{|t| \leqslant \frac{s}{c_1}} F(2,t) = \frac{s}{c_1}, \qquad \sup_{t \in \mathbb{R}} F(1,t) = +\infty,$$

so condition (6.2) is satisfied while (6.3) is not.

Now, direct computation shows that for $\lambda = -\frac{3}{2}$ the system (10) admits no solutions, while for $\lambda \neq -\frac{3}{2}$ (in particular, for every $\lambda > 0$) it has exactly one solution given by

$$x(1) = \frac{\lambda}{3 + 2\lambda}, \qquad x(2) = \frac{2\lambda + \lambda^2}{3 + 2\lambda};$$

thus, the thesis of Theorem 6 does not hold.

Example 9. Let $p, p_1 > 1$ be real numbers and consider the system

$$\begin{cases} -\varphi_p\big(\Delta x(1)\big) + \varphi_p\big(\Delta x(0)\big) = -\lambda \varphi_{p_1}\big(x(1)\big), \\ -\varphi_p\big(\Delta x(2)\big) + \varphi_p\big(\Delta x(1)\big) = 0, \\ x(0) = x(3) = 0, \end{cases} \tag{11}$$

which is of the type (P_λ) for $T = 2$ and

$$f(1,t) = -\varphi_{p_1}(t), \qquad f(2,t) = 0.$$

We have then

$$F(1,t) = -\frac{|t|^{p_1}}{p_1}, \qquad F(2,t) = 0.$$

Note that in this case

$$c_1 = \big(1 - 2^{1-p}\big)^{\frac{1}{p}}, \qquad c_2 = \big(2 + 2^p\big)^{\frac{1}{p}}.$$

As above, condition (6.1) is satisfied; besides, for arbitrary $0 < r < s$ we have

$$\sup_{\frac{r}{c_2} \leqslant |t| \leqslant \frac{s}{c_1}} F(1,t) = -\frac{r^{p_1}}{p_1 c_2^{p_1}}, \qquad \sup_{|t| \leqslant \frac{s}{c_1}} F(1,t) = \sup_{t \in \mathbb{R}} F(1,t) = 0$$

and obviously

$$\sup_{\frac{r}{c_2} \leqslant |t| \leqslant \frac{s}{c_1}} F(2,t) = \sup_{|t| \leqslant \frac{s}{c_1}} F(2,t) = \sup_{t \in \mathbb{R}} F(2,t) = 0,$$

so condition (6.3) is satisfied while (6.2) is not.

In order to study the solution set of (11), we observe that the inverse mapping of φ_p is φ_q, where $q = \frac{p}{p-1}$; hence, from the second equation of (11) we get

$$x(2) - x(1) = \varphi_q\big(\varphi_p\big(-x(2)\big)\big) = -x(2),$$

so

$$x(2) = \frac{x(1)}{2}$$

and from the first equation of (11)

$$-\left(1 + 2^{1-p}\right)\varphi_p\left(x(1)\right) = \lambda \varphi_{p_1}\left(x(1)\right). \tag{12}$$

We remark that $x(1) = 0$ always solves (12), then we distinguish three cases:

- if $p_1 < p$, from (12) we deduce that for $\lambda \geqslant 0$ (11) admits only the zero solution, while for $\lambda < 0$ it has also two nontrivial solutions given by

$$x(1) = \pm\left(-\frac{\lambda}{1 + 2^{1-p}}\right)^{\frac{1}{p-p_1}}, \qquad x(2) = \pm\frac{1}{2}\left(-\frac{\lambda}{1 + 2^{1-p}}\right)^{\frac{1}{p-p_1}}$$

  whose norms tend to $+\infty$ as $\lambda \to -\infty$;
- if $p_1 = p$, the system has a unique (negative) eigenvalue $\tilde{\lambda} = -(1 + 2^{1-p})$ such that for $\lambda = \tilde{\lambda}$ (11) admits infinitely many solutions given by $x(1) = h$, $x(2) = \frac{h}{2}$ for every $h \in \mathbb{R}$, while for $\lambda \neq \tilde{\lambda}$ (11) admits only the zero solution;
- if $p_1 > p$, from (12) we deduce that for $\lambda \geqslant 0$ (11) admits only the zero solution, while for $\lambda < 0$ it has also two nontrivial solutions given by

$$x(1) = \pm\left(-\frac{1 + 2^{1-p}}{\lambda}\right)^{\frac{1}{p_1-p}}, \qquad x(2) = \pm\frac{1}{2}\left(-\frac{1 + 2^{1-p}}{\lambda}\right)^{\frac{1}{p_1-p}}$$

  whose norms tend to 0 as $\lambda \to -\infty$.

In any case, for every $\lambda > 0$ the system (11) has only the zero solution, so the thesis of Theorem 6 does not hold.

Remark 10. In [12], Ricceri posed a question which we can rephrase as follows: can we find X, J, $\bar{x}$, p, r, s as in Theorem 3, satisfying the assumptions (3.1), (3.2) and (3.3), such that there exists a *unique* $\lambda^* > 0$ for which the functional

$$x \mapsto \frac{\|x - \bar{x}\|^p}{p} + \lambda J(x)$$

admits at least three critical points?

The problem is well motivated (see [12, Remarks 1 and 3]) but still unsolved: hopefully, our extension of Ricceri's result from the case $p = 2$ to arbitrary $p > 1$ could make it easier to find a solution (for instance in the framework of Section 3).

Acknowledgment

We wish to thank Professor B. Ricceri for his valuable remarks and discussion during the preparation of the paper.

References

[1] R.P. Agarwal, Difference Equations and Inequalities, Marcel Dekker Inc., 2000.
[2] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, Nonlinear Anal. 58 (2004) 69–73.
[3] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Difference Equ. 2005 (2005) 93–99.
[4] X. Cai, Z. Guo, J. Yu, Periodic solutions of a class of nonlinear difference equations via critical point method, Comput. Math. Appl. 52 (2006) 1639–1647.
[5] X. Cai, J. Yu, Existence theorems for second-order discrete boundary value problems, J. Math. Anal. Appl. 320 (2006) 649–661.
[6] F. Faraci, A. Iannizzotto, Multiplicity theorems for discrete boundary value problems, Aequationes Math. 74 (2007) 111–118.
[7] Z. Guo, M. Ma, Homoclinic orbits and subharmonics for nonlinear second order difference equations, Nonlinear Anal. 67 (2007) 1737–1745.
[8] L. Jiang, Z. Zhou, Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations, Adv. Difference Equ. 2008 (2008), Article ID 345916, 10 pp.
[9] V. Lakshmikantham, D. Trigiante, Theory of Difference Equations: Numerical Methods and Applications, Academic Press, 1988.
[10] M. Mihăilescu, V. Rădulescu, S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems, J. Difference Equ. Appl., in press.
[11] B. Ricceri, Well-posedness of constrained minimization problems via saddle-points, J. Global Optim. 40 (2008) 389–397.
[12] B. Ricceri, A multiplicity theorem in $\mathbb{R}^n$, J. Convex Anal. 16 (3–4) (2009).
[13] M. Struwe, Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, 2000.